2023 Power Team Solutions
Texas A&M High School Students Contest
November 2023

Problem 1 The city map is an infinite square grid of streets: horizontal lines are streets that go in the east-west direction, and vertical lines are streets that go in the north-south direction. A car A starts at the point $(0,0)$ and turns north or east on each crossing with probability $1/2$. A car B starts at the point (n,m), $n,m>0$, and turns south or west on each crossing with probability $1/2$.

The speeds of the cars are equal and they start simultaneously. Find the probability of the event that they meet, i.e., appear at the same crossing simultaneously.

You get bonus points (up to a half of the value of the problem) if you find the simplest possible expression for the answer in terms of the binomial coefficients.

Solution Note that the meeting point (k,l) is $k+l$ blocks away from $(0,0)$ and $(n+m-k-l)$ blocks away from (n,m). Thus we must have $k+l=(n+m)/2$;. If $n+m$ is odd, this is not an integer and thus the probability is zero. Assume that $n+m$ is even. Assume that $n \geq m$; possible meeting points (k,l) satisfy $k+l=(n+m)/2$ where k is in the range $(n-m)/2 \leq k \leq (n+m)/2$.

The number of possible routes from $(0,0)$ to (k,l) for the car A is $\binom{k+l}{k} = \binom{(n+m)/2}{k}$. The number of possible routes for the car B is $\binom{n+m-k-l}{n-k} = \binom{(n+m)/2}{n-k}$. The total number of routes of length $(n+m)/2$ for each car is $2^{(n+m)/2}$. So the probability for the cars to meet at (k,l) is

$$2^{-(n+m)}\binom{(n+m)/2}{k}\binom{(n+m)/2}{n-k},$$

and the total probability to meet somewhere on the diagonal $k+l=(n+m)/2$ is

$$2^{-(n+m)}\sum_{k=(n-m)/2}^{k=(n+m)/2}\binom{(n+m)/2}{k}\binom{(n+m)/2}{n-k}.$$

We can simplify this formula in the following way. Consider all possible routes that go north-east and join $(0,0)$ to (n,m). We have $\binom{n+m}{n}$ such routes, and each passes through one of the points with $k+l=(n+m)/2$. So we can turn each route into a pair of routes that join $(0,0)$ and (n,m) to (k,l). Vice versa, if we have a pair of routes for the cars that meet at (k,l), we can unite them into a single route from $(0,0)$ to (n,m). Thus we have

$$\sum_{k=(n-m)/2}^{k=(n+m)/2} \binom{(n+m)/2}{k}\binom{(n+m)/2}{n-k} = \binom{n+m}{n}.$$

Answer: $2^{-(n+m)}\binom{n+m}{n}$ if $n+m$ is even; 0 otherwise.

Problem 2 The town map is a grid of streets $3m \times m$ (there are $m+1$ streets that go in the east-west direction, and $3m+1$ streets that go in the north-south direction). Two cars A and B start at the points $(0,0)$ and $(3m,m)$, respectively, and move in the same way as in the previous problem, with one additional rule: if the car A reaches the northern edge of the grid, it turns to the east and continues to the east (without additional turns); if the car B reaches the southern edge, it turns to the west and continues to the west ((without additional turns).

Find the probability of the event that they meet.

You get bonus points (up to a half of the value of the problem) if you find the simplest possible expression for the answer in terms of the binomial coefficients.

Solution Below we will use the well-known identities for binomial coefficients, $\binom{n}{k}=\binom{n}{n-k}$ and $\sum_{k=0}^{n}\binom{n}{k}=2^n$.

Similarly to the previous item, the cars will meet at a point (k,l) with $k+l=(3m+m)/2=2m$, $m \le k \le 2m$. Let the car A travel $2m$ blocks, and let us compute the probability for the car A to get from 0 to (k,l). If $(k,l) \neq (m,m)$, the car does not reach the top or the rightmost edge of the grid on its way, thus the probability is the same as in the previous item [1], $p_{k,l}=2^{-2m}\binom{2m}{k}$. Since the total probability is 1, the probability for the car

[1]Note that although the number of admissible paths in this problem is smaller than in Proplem 1, different paths do not have the same probability: the probabilities of paths that slide along the edge of the grid are bigger. So, each path is counted according to his weight, and the number of paths of length $2m$, counting the weights, is still 2^{-2m} in the considered case of $(k,l) \neq (m,m)$

A to get to the remaining intersection (m,m) is

$$p_{m,m} = 1 - \sum_{k=m+1}^{2m} p_{k,2m-k} = 1 - \sum_{k=m+1}^{2m} 2^{-2m}\binom{2m}{k} = \sum_{k=0}^{m} 2^{-2m}\binom{2m}{k} =$$
$$2^{-2m}\binom{2m}{m} + \sum_{k=0}^{m-1} 2^{-2m}\binom{2m}{k}.$$

We separated away the first summand that is the same as in the previous item; the rest of the sum is a new addition.

Similarly, the probability for the car B to get from $(m,3m)$ to (k,l) is $q_{k,l} = 2^{-2m}\binom{2m}{3m-k}$ if $(k,l) \neq (2m,0)$; the probability to get to $(2m,0)$ is $q_{2m,0} = p_{m,m}$, due to the symmetry of the picture with respect to the point $(3m/2, m/2)$.

Thus the probability that the cars meet is

$$\sum_{k=m}^{k=2m} p_{k,2m-k} q_{k,2m-k} =$$
$$2^{-4m} \sum_{k=m}^{k=2m} \binom{2m}{k}\binom{2m}{3m-k} + \sum_{k=0}^{m-1} 2^{-2m}\binom{2m}{k} \cdot q_{m,m} + \sum_{k=0}^{m-1} 2^{-2m}\binom{2m}{k} \cdot p_{2m,0}.$$

The first part is same as in the previous item for $n = 3m$, and is simplified in the same way as before. The last two summands are new. Since $q_{m,m} = p_{2m,0} = 2^{-2m}$, the last two summands equal

$$2^{-4m}\left(\sum_{k=0}^{m-1}\binom{2m}{k} + \sum_{k=0}^{m-1}\binom{2m}{k}\right) = 2^{-4m}\left(2^{2m} - \binom{2m}{m}\right) = 2^{-2m} - 2^{-4m}\binom{2m}{m}.$$

Answer:
$$2^{-4m}\binom{4m}{m} + 2^{-2m} - 2^{-4m}\binom{2m}{m}.$$

In all problems below the taxicab distance between points A, B with coordinates (x_1, y_1) and (x_2, y_2) is $d(A,B) = |x_1 - x_2| + |y_1 - y_2|$.

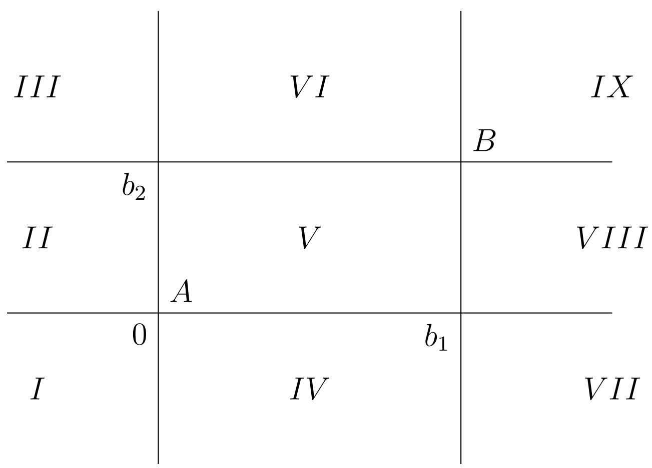
Problem 3 Suppose that the points $A = (0,0), B = (b_1, b_2)$ satisfy $b_1, b_2 \geq 0$. Describe the locus of points C such that $d(A,C) = d(C,B)$ (the "taxicab perpendicular bisector" to AB).

Solution The condition for the point $C = (x,y)$ is $|x| + |y| = |x - b_1| + |y - b_2|$. We have the following cases.

Degenerate case: If $b_1 = 0$ and $b_2 = 0$, C is an arbitrary point on the plane. If $b_1 = 0$ and $b_2 > 0$, i.e., AB is vertical, we get that x is arbitrary and $|y| = |y - b_2|$, i.e., $y = b_2/2$: C is located on the perpendicular bisector to AB. The case of horizontal AB is analogous.

Non-degenerate case: Suppose that AB is not horizontal or vertical. Assume $b_2 \leq b_1$, i.e., the angle between AB and the horizontal line is not greater than 45°. If this is not true, we can reflect the configuration over $x = y$.

The condition on the point $C = (x,y)$ is given by $|x| + |y| = |x - b_1| + |y - b_2|$.



Divide the plane into nine pieces as shown, by the lines $x = 0, y = 0, x = b_1, y = b_2$. In each of the nine pieces, we get:

I If $x \leq 0$, $y \leq 0$: $-x - y = b_1 - x + b_2 - y$ is equivalent to $b_1 + b_2 = 0$ which is not true.

II If $x \leq 0, 0 < y < b_2$: $-x + y = b_1 - x + b_2 - y$ is equivalent to $y = \frac{b_1+b_2}{2}$. However, this cannot be less than b_2 since $b_2 \leq b_1$. Thus C cannot be in this domain.

III If $x \leq 0, y \geq b_2$: $-x+y = b_1 - x + y - b_2$ is equivalent to $b_1 = b_2$. So for $b_1 = b_2$, the point C can be anywhere in this domain, and if $b_1 \neq b_2$, no points C can be here.

IV If $0 < x \leq b_1, y \leq 0$: $x - y = b_1 - x + b_2 - y$ is equivalent to $x = \frac{1}{2}(b_1 + b_2)$. This is, indeed, smaller than b_1 since $b_2 \leq b_1$. So the set of possible points C is the vertical ray $x = \frac{1}{2}(b_1 + b_2), y \leq 0$.

V If $0 < x \leq b_1$, $0 < y \leq b_2$: $x + y = b_1 - x + b_2 - y$, i.e., $x + y = \frac{1}{2}(b_1 + b_2)$. This is the line with slope -1 that passes through the midpoint $(b_1/2, b_2/2)$ of AB. Note that this line joins the endpoints of the rays from domain IV and domain VI (see below).

VI If $0 < x \leq b_1$, $y > b_2$: $x + y = b_1 - x + y - b_2$ is equivalent to $x = \frac{1}{2}(b_1 - b_2)$. The set of possible C is the ray $x = \frac{1}{2}(b_1 - b_2), y > b_2$.

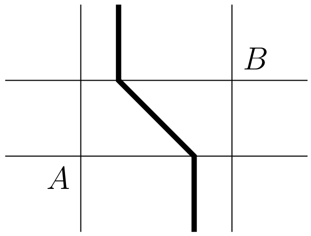
VII If $x > b_1, y \leq 0$: $x - y = x - b_1 + b_2 - y$ is equivalent to $b_1 = b_2$. So for $b_1 = b_2$, the point C can be anywhere in this domain, and if $b_1 \neq b_2$, no points C can be here.

VIII If $x > b_1, 0 < y < b_2$: $x+y = x - b_1 + b_2 - y$ is equivalent to $y = \frac{1}{2}(b_2 - b_1)$, but this expression is non-positive since $b_1 \geq b_2$. So no points C can be here.

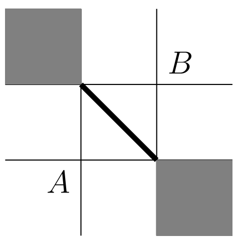
IX If $x > b_1, y \geq b_2$: $x + y = x - b_1 + y - b_2$ is equivalent to $b_1 + b_2 = 0$ which is not true.

Answer:

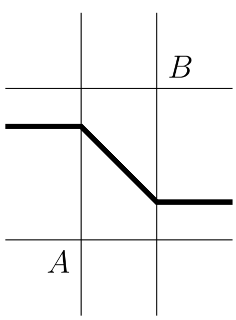
- If $b_1 = b_2 = 0$, C is arbitrary.
- If $b_1 = 0, b_2 > 0$ or $b_1 > 0, b_2 = 0$, C belongs to the perpendicular bisector of AB.
- For $b_1 > b_2$, the set of possible points C is formed by two rays $x = \frac{1}{2}(b_1 + b_2), y \leq 0$ and $x = \frac{1}{2}(b_1 - b_2), y \geq b_2$, and the segment that joins their endpoints $x + y = \frac{1}{2}(b_1 + b_2), 0 < x < b_1, 0 < y < b_2$.



- For $b_1 = b_2$, the set of possible points C is formed by two quadrants $x \leq 0, y \geq b_2$ and $x \geq b_1, y \leq 0$ and the segment $x + y = \frac{1}{2}(b_1 + b_2)$ that joins their corners. Note that in this case, the rays from domains IV and VI are located on the borders of these quadrants;



- For $b_1 < b_2$, the case is reduced to the case $b_1 > b_2$ by swapping x and y. So the set of possible points C is formed by two rays $y = \frac{1}{2}(b_1+b_2), x \leq 0$ and $y = \frac{1}{2}(b_2 - b_1), x \geq b_1$, and the segment that joins their endpoints $x + y = \frac{1}{2}(b_1 + b_2), 0 < x < b_1, 0 < y < b_2$.

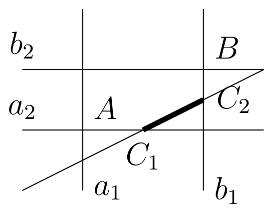


Problem 4 Given two points A, B above the line $y = kx, 0 < k < 1$, find all points C on the line $y = kx$ such that the distance $d(A, C) + d(B, C)$ is as small as possible.

Solution Let $A = (a_1, a_2)$, $B = (b_1, b_2)$; assume that $a_2 \leq b_2$, otherwise we swap A and B.

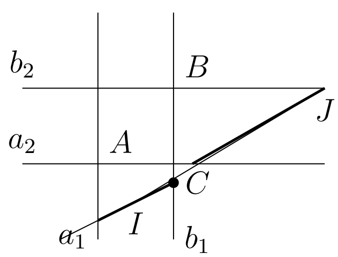
Consider the lines $x = a_1, x = b_1,\ y = a_2, y = b_2$. The pictures show possible cases for the location of the line $y = kx$ with respect to these lines, namely:

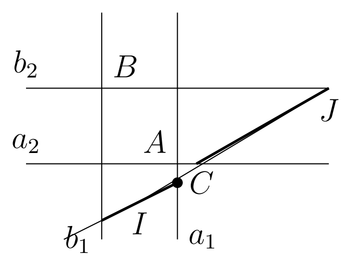
Case 1: $a_1 \le b_1$, $a_2 \le kb_1$. Then the line $y = kx$ intersects the rectangle $a_1 \le x \le b_1$, $a_2 \le x \le b_2$ over the segment $[C_1, C_2]$ with $C_1 = (a_2/k, a_2)$ and $C_2 = (b_1, kb_1)$.



In this rectangle, $\mathrm{dist}(C, A) + \mathrm{dist}(C, B) = |x - a_1| + |y - a_2| + |x - b_1| + |y - b_2| = (x - a_1) + (y - a_2) + (b_1 - x) + (b_2 - y) = \mathrm{dist}(A, B)$. Outside this rectangle, one of the inequalities $|x - a_1| + |x - b_1| \ge |a_1 - b_1|$ or $|y - a_2| + |y - b_2| \ge |a_2 - b_2|$ is strict, thus $\mathrm{dist}(C, A) + \mathrm{dist}(C, B) > \mathrm{dist}(A, B)$. Thus the set of points on $y = kx$ with the minimum possible total distance to A, B is the interval $[C_1, C_2]$.

Case 2: either $a_2/k > b_1 \ge a_1$ or $a_1 > b_1$ (in the latter case $a_2 > ka_1$ since A is above $y = kx$ and thus we have $a_2/k > a_1 > b_1$). Here the line $y = kx$ does not intersect this rectangle.





Let $C = (x, kx)$ and consider the function $\mathrm{dist}(C, A) + \mathrm{dist}(C, B) = |x - a_1| + |kx - a_2| + |b_1 - x| + |b_2 - kx|$.

Note that the function $|x - a_1| + |b_1 - x|$ decreases to the left from the interval $I = [a_1, b_1]$ (or $[b_1, a_1]$ if $b_1 > a_1$), is constant on I, and increases to the right from I. The function $|kx - a_2| + |b_2 - kx|$ decreases to the left from the interval $J = [a_2/k, b_2/k]$, is constant on J, and increases to the right from J. The interval I is to the left from the interval J and does not intersect it since $a_2/k > \max(a_1, b_1)$. Thus the sum of these functions will decrease for $x < max(a_1, b_1)$ (where either both functions decrease or one is constant and the other decreases), and similarly, the sum increases for $x > a_2/k$. We conclude that the minimum of the function must be on the interval $\max(a_1, b_1) \le x \le a_2/k$ between I and J.

On this interval, $\text{dist}(C, A) + \text{dist}(C, B) = 2x - a_1 - b_1 + a_2 + b_2 - 2kx$. This expression grows with x since $0 < k < 1$, thus it takes its minimum at the leftmost point $x = \max(a_1, b_1), y = kx$.

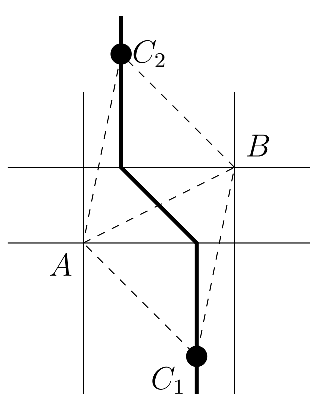
Answer: if $a_2/k \le b_1$ — the interval $a_2/k \le x \le b_1, y = kx$;

if $a_2/k > \max(a_1, b_1)$, the point $C = (\max(a_1, b_1), k \max(a_1, b_1))$.

Problem 5 Suppose that a triangle ABC is taxicab equilateral: $d(A, B) = d(B, C) = d(C, A)$. Show that one of the sides of ABC is vertical, horizontal, or has a slope ± 1.

Solution Suppose that the side AB is not horizontal or vertical and does not have slope ± 1. Let $A = (0, 0)$ and assume that $B = (b_1, b_2)$ satisfies $b_1, b_2 > 0$ and $b_1 > b_2$. We can always achieve this by placing the origin at the leftmost vertex A and relecting the configuration with respect to $x = 0, y = 0$, and $x = y$.

Then C is located on the perpendicular bisector to the AB, namely on the union of the rays $x = \frac{1}{2}(b_1 + b_2), y < 0$, $x = \frac{1}{2}(b_1 - b_2), y > b_2$ and a segment $x + y = \frac{1}{2}(b_1 + b_2), 0 < x < b_1, 0 < y < b_2$. On this segment, the taxicab distance to A and B is $\frac{1}{2}(b_1 + b_2)$; since the distance $\text{dist}(A, C)$ should be equal to $\text{dist}(A, B) = b_1 + b_2$, C cannot be on this segment and must be on one of the rays. We find that C is either $C_1 = (\frac{1}{2}(b_1 + b_2), -\frac{1}{2}(b_1 + b_2))$ or $C_2 = (\frac{1}{2}(b_1 - b_2), \frac{1}{2}(b_1 + 3b_2))$. In both cases, AC or BC has slope ± 1.



Problem 6 For a taxicab equilateral triangle ABC,

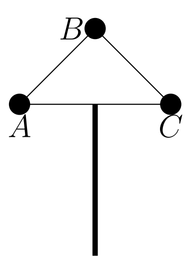
1. prove that there always exists a point O such that $d(O, A) = d(O, B) = d(O, C)$, i.e., we can always inscribe an equilateral triangle in a taxicab circle;

2. Provide an example of a taxicab equilateral triangle such that a point O with this property is not unique.

Solution (contains simultaneously solutions of both items) Special case. Suppose that all segments AB, BC, AC are horizontal, vertical, or have slopes ± 1. Thus ABC is a right triangle with sides parallel to $x = 0, y = 0, x = y$.

If only one side has a slope ± 1, the triangle is not equilateral. E.g., if AB is horizontal and AC is vertical, with $A = (0, 0)$, $B = (b, 0)$, $C = (0, b)$, the distance $\text{dist}(B, C)$ is $2b$ and not b.

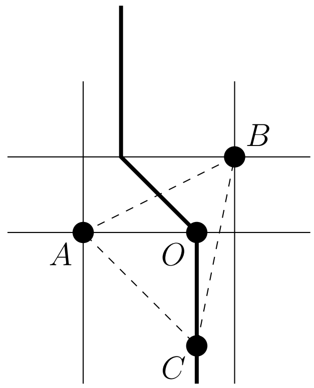
The only possible case is when both AB and BC have slopes ± 1. Reflecting over $x = \pm y$ if necessary, we will assume that $A = (-a, 0)$, $B = (0, a)$, $C = (a, 0)$. Then any point with $y = 0, x \leq 0$ satisfies $\text{dist}(A, O) = \text{dist}(B, O) = \text{dist}(C, O) = |x| + a$, thus there are infinitely many points O with required property. **This proves that O exists and provides an example when it is non-unique, so completes item (b).**



General case. Suppose that the triangle ABC has a side that is not vertical, not horizontal, and its slope is not ± 1. Let this side be AB. As in the previous problem, set $A = (0, 0)$ and $B = (b_1, b_2)$ with $b_1 > b_2$; we have determined that C should be $C_1 = (\frac{1}{2}(b_1 + b_2), -\frac{1}{2}(b_1 + b_2))$ or $C_2 = (\frac{1}{2}(b_1 - b_2), \frac{1}{2}(b_1 + 3b_2))$. The two points are symmetric with respect to the middle of AB, thus it is sufficient to consider the first case $C = (\frac{1}{2}(b_1 + b_2), -\frac{1}{2}(b_1 + b_2))$.

It is easy to see that the point $O = (\frac{1}{2}(b_1 + b_2), 0)$ will satisfy the equalities $\text{dist}(A, O) = \text{dist}(B, O) = \text{dist}(C, O)$. Thus such point exists. One can check that it is unique in this case, but this was not required in the problem.

Thus in the non-degenerate case, the point O exists and is unique.



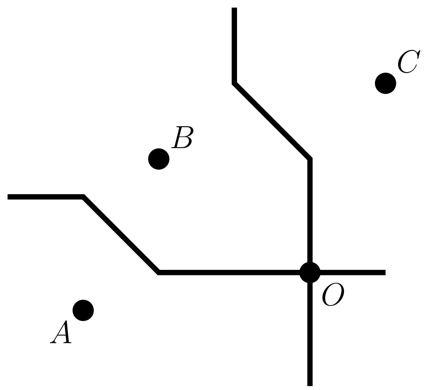
Problem 7 Describe all triangles ABC such that we cannot inscribe ABC in a taxicab circle, i.e., there is no point O with the property $d(O, A) = d(O, B) = d(O, C)$.

Solution Call a segment AB "almost horizontal" if its slope is between -1 and 1, and "almost vertical" otherwise.

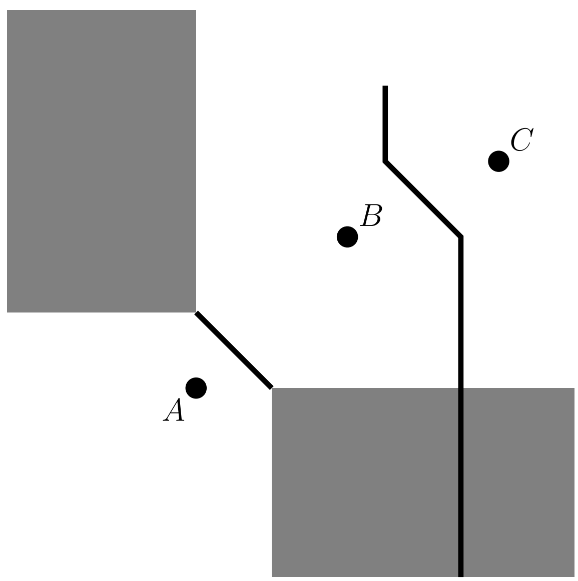
Answer: if all segments AB, BC, AC are almost vertical, then O does not exist. If all these segments are almost horizontal, then O does not exist as well. In all other cases, O exists.

Indeed, Problem 3 implies that the taxicab perpendicular bisector of any almost horizontal segment AB is formed by two vertical rays and the slanted segment that joins their endpoints (this slanted segment degenerates into a point if AB is horizontal). Similarly, the taxicab perpendicular bisector of any almost vertical segment is formed by two horizontal rays and the slanted segment that joins their endpoints.

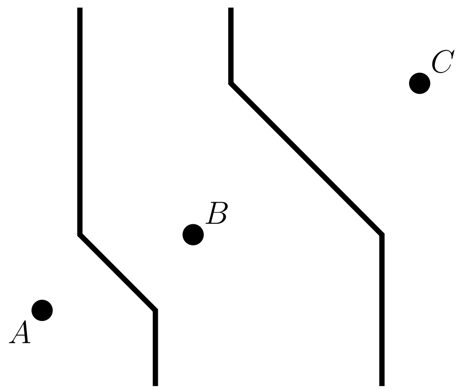
So if AB is almost vertical and BC is almost horizontal, then their taxicab perpendicular bisectors must intersect. The intersection point O satisfies $\text{dist}(A, O) = \text{dist}(B, O) = \text{dist}(C, O)$.



If AB, BC, or AC has slope exactly 1 or -1, the corresponding taxicab perpendicular bisector is a union of two quadrants and the segment that joins their corners. This set intersects all possible taxicab perpendicular bisectors. In this case, O will also exist.



Finally, assume that all segments AB, BC, AC are almost horizontal. Assume that $A = (0, 0)$ is the leftmost point, $B = (b_1, b_2)$, and $C = (c_1, c_2)$ is the rightmost point. Then the taxicab perpendicular bisector of AB is contained in the strip $0 < x < b_1$, and the taxicab perpendicular bisector of BC is a contained in the strip $b_1 < x < c_1$. Thus they do not intersect, and there cannot be a point O with $\text{dist}(A, O) = \text{dist}(B, O) = \text{dist}(C, O)$.



The case when all segments AB, BC, AC are almost vertical is considered in an analogous way.

Problem 8 Points $A_1, \dots, A_n$, $n \geq 4$, satisfy the following: $1 = d(A_1, A_2) = d(A_2, A_3) = \dots = d(A_n, A_1)$, and there exists a point O such that $d(O, A_1) = d(O, A_2) = \dots = d(O, A_n) = r$. Some of the points A_k may coincide. For every n, find the minimal possible value of r.

Solution

Answer: $r = 1/2$

Note that for the taxicab distance, we have a version of the triangle inequality: $\operatorname{dist}(A, B) + \operatorname{dist}(B, C) \geq \operatorname{dist}(A, C)$. Indeed, we have for $A = (x_A, y_A)$, $B = (x_B, y_B)$, $C = (x_C, y_C)$:

$$|x_A - x_B| + |y_A - y_B| + |x_B - x_C| + |y_B - y_C| \geq |x_A - x_C| + |y_A - y_C|$$

due to the inequality $|x_A - x_B| + |x_B - x_C| \geq |x_A - x_C|$ and an analogous one for y.

Thus $1 = d(A_1, A_2) \leq d(O, A_1) + d(O, A_2) = 2r$, i.e., $r \geq 1/2$.

To see that $r = 1/2$ is possible, we will construct an example, separately for odd and for even n.

For even $n = 2k$, take $A_1 = A_3 = \dots A_{2k-1} = (0.5, 0)$, and $A_2 = A_4 = \dots A_{2k} = (-0.5, 0)$, and $O = (0, 0)$. It is easy to see that all conditions hold true.

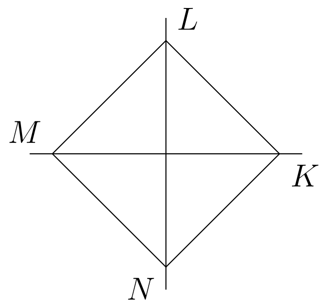
For odd $n = 2k + 1$, take $A_1 = (0, 0.5)$, $A_2 = A_4 = \dots A_{2k} = (-0.5, 1)$, $A_3 = \dots A_{2k+1} = (0.5, 0)$, and $O = (0, 0)$. Since $\operatorname{dist}(A_1, A_2) = \operatorname{dist}(A_{2k+1}, A_1) = 1$, all conditions still hold true.

Problem 9 For distinct points $A_1, \dots, A_n$, $n \geq 4$, assume that the polygon $A_1 \dots A_n$ is non-self-intersecting and satisfies the same requirements as in Problem 8 . For every n, find the minimal possible value of r.

Solution

Answer: for $n = 4m$, we have $r = m/2 = n/8$. For $n = 4m + k, k = 1, 2, 3$, we have $r = (m + 1)/2 = (\lfloor n/4 \rfloor + 1)/2$.

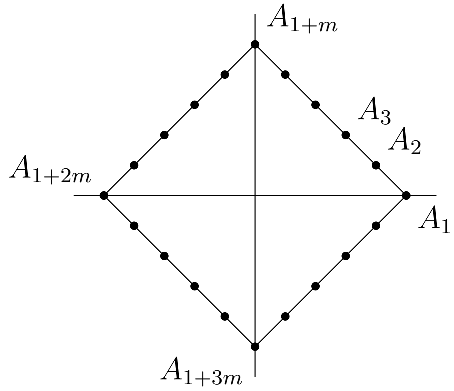
Consider the set of points X such that $\text{dist}(O, X) = 1$ (the taxicab circle of radius 1). This set is given by $|x| + |y| = 1$, and is thus a square with sides $x + y = 1, x > 0, y > 0$; $-x + y = 1, x < 0, y > 0$; $x - y = 1, x > 0, y < 0$; $x + y = -1, x < 0, y < 0$. Note that taxicab lengths of these segments are equal to $2r$. Let K, L, M, N be vertices of this square. All points $A_1, A_2, \dots, A_n$ are located on this taxicab circle, and since $A_1 \dots A_n$ is non-self-intersecting, they are numbered along the circle (either clock- or counterclockwise).



Put $n = 4m + k$, $k = 0, 1, 2, 3$.

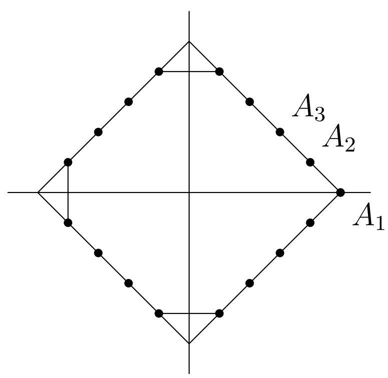
Suppose that k=0. Applying the triangle inequality, we find that $8r = \text{dist}(K, L) + \text{dist}(L, M) + \text{dist}(M, N) + \text{dist}(N, K) \geq \text{dist}(A_1, A_2) + \text{dist}(A_2, A_3) + \dots + \text{dist}(A_n, A_1) = n$, thus $r \geq n/8$.

To see that $r = n/8$ is possible, place four vertices $A_1, A_{1+m}, A_{1+2m}, A_{1+3m}$ of the polygon at the vertices K, L, M, N of the taxicab circle of radius $r = n/8$ and distribute other points evenly along the taxicab circle. Then we have $2r = \text{dist}(A_1, A_{1+m}) = m = n/4$ and $r = n/8$.



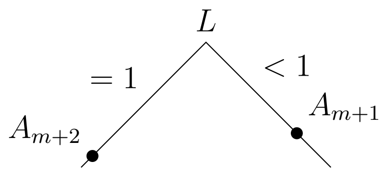
Suppose that $k \neq 0$**,** $n > 4$.

Then we can achieve $r = (\lfloor n/4 \rfloor + 1)/2 = (m+1)/2$. To see this, place $4m+4$ points $A_1, \ldots, A_{4m+4}$ along the taxicab circle of radius $r = (m+1)/2$ in the same way as in the previous case, and then remove $(4-k)$ of them from the vertices of the taxicab circle. The remaining $4m+k$ points will satisfy the assumptions. Indeed, the points near the vertex of the taxicab circle have coordinates $(r-0.5, -0.5)$, $(r, 0)$, $(r-0.5, 0.5)$, and all pairwise taxicab distances between them are 1, so the conditions on the points $A_1, \ldots, A_n$ will be still satisfied if we remove the vertex from $(r, 0)$.



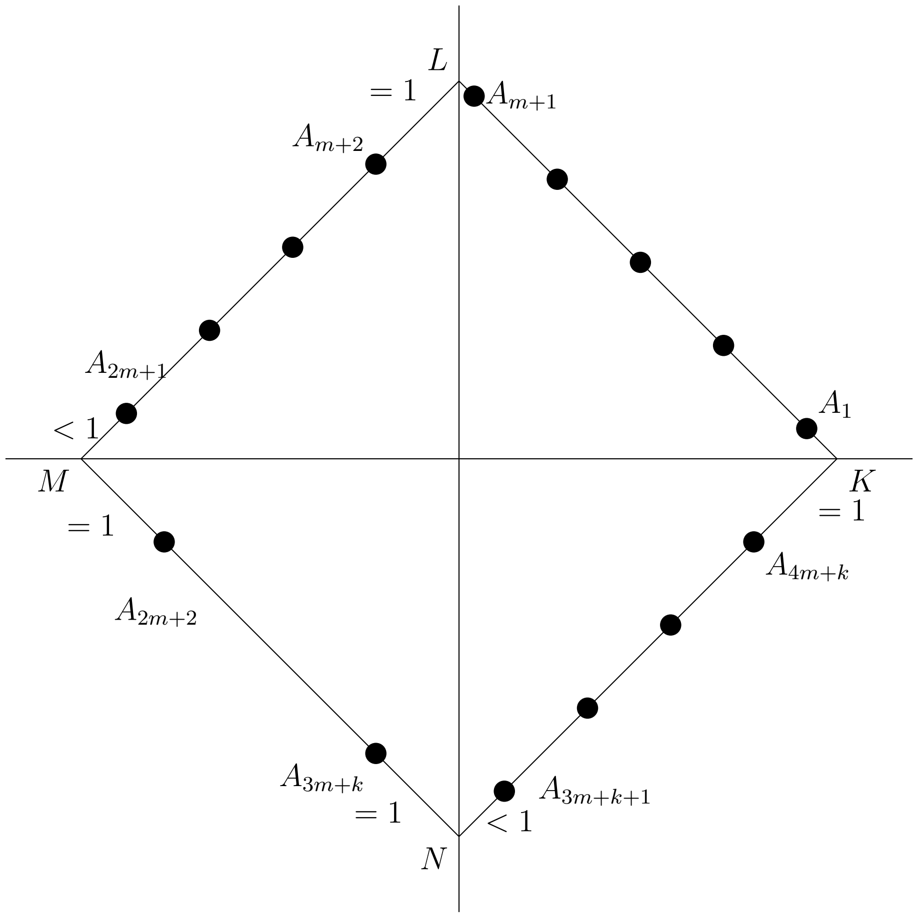
Suppose that a smaller value of r is possible, and let us show that $n = 4m+k$ points $A_1, \ldots, A_n$ at distances 1 cannot fit along the taxicab circle of radius r. Since $2r < m+1$, triangle inequality implies that the side of the taxicab circle (of taxicab length $2r$) cannot contain $m+2$ or more points A_j. On the other hand, at least one side of the taxicab circle must contain at least $m+1$ points A_j since there are $4m+k$ points in total. Consider the side KL that contains exactly $m+1$ points $A_1, \ldots, A_{m+1}$. Since $2r < m+1$, the taxicab distances $\text{dist}(K, A_1)$ and $\text{dist}(A_{m+1}, L)$ are smaller than 1.

Now, show that $\text{dist}(A_{m+1}, L) < 1$ implies $\text{dist}(L, A_{m+2}) = 1$. Indeed, suppose that $L = (0, r)$. We have $A_{m+1} = (a, r - a)$ with $a < 0.5$. Then $A_{m+2} = (-b, r - b)$ satisfies $\text{dist}(A_{m+1}, A_{m+2}) = 1 = a + b + |b - a|$. If $b \leq a$, this is at most $a + b + a - b = 2a < 1$ and we get a contradiction. If $b > a$, we have $1 = a + b + b - a$ and thus $b = 1/2$, $\text{dist}(L, A_{m+2}) = 1$.



Since $2r < m+1$, the side LM contains at most m vertices of the polygon (not counting L if $A_{m+1} = L$).

Similarly, $\text{dist}(A_1, K) < 1$ implies that $\text{dist}(K, A_n) = 1$ and the side NK contains at most m vertices of the polygon (not counting K if $A_1 = K$) .



If $2r < m$, both LM and NK contain strictly less than m vertices, then the taxicab circle contains at most $(m+1)+(m-1)+(m-1)+(m+1) = 4m$ vertices and we get a contradiction.

Suppose that $m \leq 2r < m+1$: then LM and NK contain exactly m vertices of the polygon, namely $A_{m+2}, \ldots, A_{2m+1}$ and $A_{4m+k}, A_{4m+k-1}, \ldots, A_{3m+k+1}$. Moreover, since $\mathrm{dist}(L, A_{m+1}) = 1$, we have $\mathrm{dist}(A_{2m+1}, M) < 1$. This again implies that $\mathrm{dist}(M, A_{2m+2}) = 1$. Similarly, since $\mathrm{dist}(K, A_{4m+k}) = 1$, we have $\mathrm{dist}(A_{3m+k+1}, N) < 1$ and thus $\mathrm{dist}(N, A_{3m+k}) = 1$.

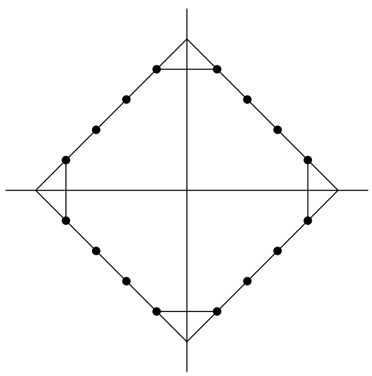
This is not possible: since A_{2m+2} and A_{3m+k} are located on the same side MN at taxicab distances 1 from its endpoints and the taxicab distance between A_{2m+2} and A_{3m+k} is $m+k-2$, we must have $\mathrm{dist}(M, N) = 2r = m+k \geq m+1$.

Problem 10 For distinct points $A_1, \ldots, A_n$, $n \geq 4$, assume that the polygon $A_1 \ldots A_n$ is non-self-intersecting and satisfies the same requirements as in Problem 8 . For every n, find the maximal possible value of r.

Solution

Answer: If $n = 4m + k$ with $k = 0, 1, 2, 3$, then $r = (m + 1)/2$.

If $n = 4m+k$ with $k = 1, 2, 3$, the example from the previous item shows that $r = (m+1)/2$ is possible. If $n = 4m$, we can also achieve $r = (m+1)/2$. Namely, we can place m vertices along each side of the taxicab circle at distances $1, 2, \ldots, m$ from vertices.



Let us prove that the larger value of r is not possible.

Indeed, suppose that $n = 4m + k$, $k = 0, 1, 2,$ or 3, and $2r > (m + 1)$. Suppose that $A_1, \ldots, A_s$ are all vertices of the polygon that are located on the side KL of the taxicab circle. Then we have $\text{dist}(K, A_1) \leq 1$, otherwise the side NK contains no points at a distance 1 from A_1 and cannot contain A_n. Similarly, $\text{dist}(A_s, L) \leq 1$. Since $\text{dist}(A_1, A_s) = s - 1$, we have $s + 1 \geq 2r > m + 1$. Thus $s > m$: each side of the taxicab circle contains at least $m + 1$ vertices of the polygon.

Since there are $4m + k < 4m + 4$ vertices in total, some of the vertices of the polygon must be in K, L, M, or N. Suppose that $K = A_1$.

Since $\text{dist}(K, L) = 2r$, the sides KL and KN contain at least $m + 2$ vertices each ($K = A_1$ is counted twice here).

If L and N are not vertices of the polygon, then the total number of its vertices is at least $(m+2)+(m+2)+(m+1)+(m+1)-2 = 4m+4$ since only two vertices could be counted twice, and we get a contradiction.

If L, N are vertices of the polygon, then the sides LM, MN also contain at least $m+2$ vertices of the polygon, and the total number of vertices is at least $4m+8-4=4m+4>4m+k$. We get a contradiction.